Chapter 3:
Determinants and Diagonalization

Sec. 3.3:
Diagonalization and Eigenvalues

Eigenvalues/Eigenvectors

Def: Let A be an $n \times n$ matrix. If there is a number λ and a non-zero $n \times 1$ column matrix $\boldsymbol{x}$ such that

$$A\boldsymbol{x} = \lambda\boldsymbol{x}$$

Then $\boldsymbol{x}$ is called an eigenvector of A and λ is called an eigenvalue of A.

Ex 1: Show that $\boldsymbol{x} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ is an eigenvector of

$A = \begin{bmatrix} 4 & 5 & 3 \\ -2 & 7 & 6 \\ 0 & 1 & 5 \end{bmatrix}$. What is the eigenvalue λ corresponding

to this eigenvector?

Eigenvalues/Eigenvectors

Question: How do you find the eigenvalues and eigenvectors of a matrix?

Answer:

1. Start off by finding the eigenvalues of a matrix…

Def: If A is an $n \times n$ matrix, the characteristic polynomial of A is…

$$c_A(x) = \det(xI - A)$$

Note: The eigenvalues λ of a square matrix are the roots of its characteristic polynomial.

Def: An eigenvalues λ of a square matrix has multiplicity m if it occurs m times as a root of the characteristic polynomial.

Eigenvalues/Eigenvectors

Question: How do you find the eigenvalues and eigenvectors of a matrix?

Answer:

1. Start off by finding the eigenvalues of a matrix…

   …by finding the roots of its characteristic polynomial

2. Then, one eigenvalue λ at a time, find the eigenvectors corresponding to the eigenvalue λ

   …by finding all non-zero solutions to the homogeneous system $(\lambda I - A)\boldsymbol{x} = \mathbf{0}$

Eigenvalues/Eigenvectors

Ex 2: Find the eigenvalues, corresponding eigenvectors, and a basic set of eigenvectors the matrix $A = \begin{bmatrix} 11 & 0 & 21 \\ 0 & -1 & 0 \\ -8 & 0 & -15 \end{bmatrix}$

Eigenvalues/Eigenvectors

Ex 3: Find the eigenvalues, corresponding eigenvectors, and a basic set of eigenvectors the matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 3 & -2 \end{bmatrix}$

Diagonalization

Def: An $n \times n$ matrix D is a diagonal matrix if its only nonzero entries are on its main diagonal.

Like $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -9 & 0 \\ 0 & 0 & 7 \end{bmatrix}$

Def: An $n \times n$ matrix A is diagonalizable if there exists an invertible $n \times n$ matrix P such that…

$$D = P^{-1}AP$$

…is a diagonal matrix. P is called a diagonalizing matrix.

Like $A = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$. Use $P = \begin{bmatrix} 1 & 5 \\ -1 & 1 \end{bmatrix}$.

Then $P^{-1} = \begin{bmatrix} 1/6 & -5/6 \\ 1/6 & 1/6 \end{bmatrix}$ and $P^{-1}AP = \begin{bmatrix} -2 & 0 \\ 0 & 4 \end{bmatrix}$.

Diagonalization

<u>Question</u>: How is diagonalization related to eigenvalues and eigenvectors?

So…
If A is diagonalizable, then D is the matrix with the eigenvalues of A on its diagonal.

Diagonalization

Theorem 3.3.4

Let A be an $n \times n$ matrix.

1. *A is diagonalizable if and only if it has eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ such that the matrix $P = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \dots & \mathbf{x}_n \end{bmatrix}$ is invertible.*
2. *When this is the case, $P^{-1}AP = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ where, for each i, λ_i is the eigenvalue of A corresponding to $\mathbf{x}_i$.*

Theorem 3.3.5

A square matrix A is diagonalizable if and only if every eigenvalue λ of multiplicity m yields exactly m basic eigenvectors; that is, if and only if the general solution of the system $(\lambda I - A)\mathbf{x} = \mathbf{0}$ has exactly m parameters.

Theorem 3.3.6

An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

Diagonalization

Diagonalization Algorithm

To diagonalize an $n \times n$ matrix A:

Step 1. Find the distinct eigenvalues λ of A.

Step 2. Compute a set of basic eigenvectors corresponding to each of these eigenvalues λ as basic solutions of the homogeneous system $(\lambda I - A)\mathbf{x} = \mathbf{0}$.

Step 3. The matrix A is diagonalizable if and only if there are n basic eigenvectors in all.

Step 4. If A is diagonalizable, the $n \times n$ matrix P with these basic eigenvectors as its columns is a diagonalizing matrix for A, that is, P is invertible and $P^{-1}AP$ is diagonal.

Diagonalization

Ex 4: Diagonalize $A = \begin{bmatrix} 11 & 0 & 21 \\ 0 & -1 & 0 \\ -8 & 0 & -15 \end{bmatrix}$

Diagonalization

Ex 5: Diagonalize $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 3 & -2 \end{bmatrix}$. Then calculate A^{12}.

Diagonalization

Ex 6: Show that $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is not diagonalizable.

Diagonalization

Ex 7 (book homework #13):

If A is diagonalizable and 1 and -1 are the only eigenvalues of A, show that $A^{-1} = A$

Diagonalization

<u>Ex 8 (book homework #15)</u>:
If A is diagonalizable and $\lambda \geq 0$ for each eigenvalue of A, show that $A = B^2$ for some matrix B.

What you need to know from the book

Book reading

Pages: 173 (start at eigenvalues and eigenvectors) – 175, 178-181

Problems you need to know how to do from the book

#'s 1, 3, 6-20, 23-24, 26-27